

# Discrete flavour symmetries for degenerate solar neutrino pair and their predictions

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## Abstract

Flavour symmetries appropriate for describing a neutrino spectrum with degenerate solar pair and a third massive or massless neutrino are discussed. We demand that the required residual symmetries of the leptonic mass matrices be subgroups of some discrete symmetry group  $G_f$ .  $G_f$  can be a subgroup of  $SU(3)$  if the third neutrino is massive and we derive general results on the mixing angle predictions for various discrete subgroups of  $SU(3)$ . The main results are: (a) All the  $SU(3)$  subgroups of type C fail in simultaneously giving correct  $\theta_{13}$  and  $\theta_{23}$ . (b) All the groups of type D can predict a relation  $\cos^2 \theta_{13} \sin^2 \theta_{23} = \frac{1}{3}$  among the mixing angles which appears to be a good zeroth order approximation. Among these, various  $\Delta(6n^2)$  groups with  $n \geq 8$  can simultaneously lead also to  $\sin^2 \theta_{13}$  in agreement with global fit at  $3\sigma$ . (c) The group  $\Sigma(168) \cong PSL(2, 7)$  predicts near to the best fit value for  $\theta_{13}$  and  $\theta_{23}$  within the  $1\sigma$  range. All discrete subgroups of  $U(3)$  with order  $< 512$  and having three dimensional irreducible representation are considered as possible  $G_f$  when the third neutrino is massless. Only seven of them are shown to be viable and three of these can correctly predict  $\theta_{13}$  and/or  $\theta_{23}$ . The solar angle remains undetermined at the leading order in all the cases due to degeneracy in the masses. A class of general perturbations which can correctly reproduce all the observables are discussed in the context of several groups which offer good leading order predictions.

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## I. INTRODUCTION

The lack of direct evidence of the neutrino mass scale allows three distinct possibilities for the neutrino masses, hierarchical, quasidegenerate or partially degenerate spectrum consisting of two degenerate states forming a solar pair in zeroth order and the third massive or nearly massless state. Mass of the third state (solar pair) lies at the atmospheric scale in case of the normal (inverted) hierarchy in neutrino masses. The third neutrino can even be massless in case of the inverted hierarchy. Near degeneracy of two masses has an economical explanation in which two of the active neutrinos combine to form a Dirac state with very tiny mass difference between them. This requires special symmetries of the neutrino mass matrix and eventually of the underlying Lagrangian if this symmetry is not accidental. There exists numerous examples of continuous symmetries starting from minimal  $U(1)$ , e.g.  $L_e - L_\mu - L_\tau$  (see [1] for an exhaustive reference list) to larger groups  $O(3)_l \times O(3)_e \times O(3)_\nu \times U(1)_R$  [2] which lead to two or all three degenerate neutrinos. But discrete symmetries achieving this are not much explored except in some simple cases, e.g.  $S_3$  [3],  $A_4$  [4]. Aim of the present paper is to make an exhaustive search for discrete symmetries leading to degenerate solar pair and work out their predictions not only for the neutrino mass pattern but also for the leptonic mixing angles and CP phases.

Method of searching for possible symmetry groups  $G_f$  of an underlying theory from the symmetries of the leptonic mass matrices is discussed and studied extensively in the literature, see [5–9] for recent reviews. It is assumed that symmetries  $G_\nu$  and  $G_l$  of the neutrino and the charged lepton mass matrices respectively are not accidental but arise when  $G_f$  is spontaneously broken. In this case, if  $G_\nu$  and  $G_l$  are known a priori then the minimal group  $G_f$  which contain these as subgroups may be regarded as a symmetry of the theory. This approach makes definite prediction for the leptonic mixing matrix which solely depend on the choice of  $G_\nu$  and  $G_l$  [10–14] while dynamics is invoked to assure that  $G_f$  is broken down to the required symmetries of the leptonic mass matrices. Possible choices of  $G_f$  based on this approach and the resulting mixing patterns are studied extensively in case of the three non-degenerate massive Majorana neutrinos [15–21] and also Dirac neutrinos [22, 23]. Using a similar approach, a new class of  $G_f$  is proposed and extensively studied recently by us in case of Majorana neutrinos with a massless state [24, 25]. Our aim is to apply the same method for the description of the neutrino mass spectrum with an active Dirac pair and a massive or massless Majorana neutrino. The required residual symmetries and the nature of the resulting Dirac neutrino is quite different from the case of the conventional Dirac neutrinos studied earlier in [22, 23]. The masses of the conventional Dirac neutrinos are not naturally suppressed due to inherent lepton number conservation and one needs to invoke some additional mechanism to suppress them. In contrast, at least some of the lepton numbers  $L_e, L_\mu, L_\tau$  are broken if not all when two of the active neutrinos combine to form a Dirac state and the smallness of neutrino masses gets directly linked to lepton number violation. The earlier study of the active Dirac pair which also included the case of all degenerate neutrinos was presented in [26] but it was restricted only to the

finite von-Dyck groups which accommodates the cyclic groups, dihedral groups,  $A_4$ ,  $S_4$  and  $A_5$  [17, 18]. The following analysis goes well beyond these groups and encompasses all the discrete subgroups (DSG) of  $SU(3)$  having three dimensional irreducible representations (IR) and leads to many new predictions for mixing angles. In addition, we also explore the DSG of  $U(3)$  which can be used as the symmetry of degenerate solar pair and a massless neutrino.

In section II, we first briefly review the general approach of building  $G_f$  from the knowledge of leptonic mixing angles and then modify it to accommodate the possibility of degenerate solar pair. Following it, we analyze different DSG of  $SU(3)$  for their predictions of mixing angles in section III and provide a numerical scan of such groups in section IV. We then discuss the possibility of having a degenerate solar pair and massless third neutrino from DSG of  $U(3)$  in section V and provide some realistic examples of generating the solar mass difference in section VI. The study is finally summarized in section VII.

## II. RESIDUAL SYMMETRIES LEADING TO DEGENERATE SOLAR PAIR

Let us first briefly review the widely discussed approach [10–16] in which the knowledge of lepton mixing pattern is used to obtain the underlying leptonic symmetries. The Majorana neutrino mass matrix  $M_\nu$  with three arbitrary non-zero masses is known to be always invariant under  $Z_2 \times Z_2$  group which can be defined in an arbitrary weak basis as

$$S_1 = V_\nu \text{Diag.}(1, -1, -1) V_\nu^\dagger \quad \text{and} \quad S_2 = V_\nu \text{Diag.}(-1, 1, -1) V_\nu^\dagger, \quad (1)$$

By construction,  $S_1$  and  $S_2$  commute. Likewise, the combination  $M_l M_l^\dagger$  of the charged lepton mass matrix  $M_l$  is invariant under a  $Z_n \times Z_m \times Z_p$  symmetry

$$T_l = V_l \text{Diag.}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}) V_l^\dagger, \quad (2)$$

in an arbitrary basis with  $\phi_{e,\mu,\tau}$  being some discrete phases. The predictive power of these symmetries follow from the observation [10–16] that the neutrino mixing matrix  $U_{\text{PMNS}}$  is determined in terms of  $V_\nu$  and  $V_l$ :

$$U_{\text{PMNS}} = V_l^\dagger V_\nu, \quad (3)$$

$V_l$  and  $V_\nu$  should be different to get a non-zero mixing which means that the groups generated by  $S_i$  and  $T_l$  do not commute. Thus they cannot be imposed as a symmetry in the basic Lagrangian. However, these groups can appear as subgroups of some bigger symmetry  $G_f$  whose breaking can lead to the leptonic mass matrices invariant under respective symmetries. Extensive studies of possible  $G_f$  and the resulting mixing patterns [15–20] show that the groups which can predict all three mixing angles correctly within  $3\sigma$  are few and big [19] but one can obtain several good zeroth order mixing patterns from the smaller groups like,  $A_5$ ,  $S_4$  etc.

The diagonal parts in definition of  $S_{1,2}$  correspond to trivial symmetries of change of sign of the neutrino fields in their mass basis. This leads to two possible generalizations which will be considered in this paper. When one of the neutrinos is massless, then the phase of the corresponding mass eigenstate can be changed without affecting the underlying Lagrangian [24]. In this case, one of the  $S_i$  is replaced by

$$\tilde{S}_\nu = V_\nu \text{Diag.}(\eta, 1, -1) V_\nu^\dagger, \quad (4)$$

with  $\eta^n = 1$  and  $n \geq 3$ . The  $\tilde{S}_\nu$  forms a  $Z_n$  ( $Z_{2n}$ ) group for even (odd)  $n$ . The  $\tilde{S}_\nu$  thus represents a symmetry of a mass matrix with one massless and two non-degenerate massive Majorana neutrinos. Since  $\det(\tilde{S}_\nu) \neq \pm 1$ , the group embedding this would be a DSG of  $U(3)$  rather than of  $SU(3)$ . This idea was proposed in [24] and subsequently exhaustive search of possible DSG of  $U(3)$  was made and predictions for mixing angles were worked out in [25]. It was found that only one of the 75 DSG of  $U(3)$  having order  $< 512$  and possessing 3-dimensional IR was able to predict a massless neutrino and three mixing angles moderately close to their experimental values at the leading order.

A slight variation of the above two symmetries can be used to describe a pair of degenerate neutrinos and a third massive or massless neutrinos. Consider the following discrete symmetry in arbitrary basis

$$S_{1\nu} = V_\nu \text{Diag.}(\eta, \eta^*, 1) V_\nu^\dagger, \quad (5)$$

where  $\eta^n = 1$  and  $n \geq 3$ . The neutrino mass matrix  $M_\nu$  invariant under this symmetry, *i.e*  $S_{1\nu}^T M_\nu S_{1\nu} = M_\nu$ , can be written as

$$V_\nu^T M_\nu V_\nu = \begin{pmatrix} 0 & m & 0 \\ m & 0 & 0 \\ 0 & 0 & M \end{pmatrix}, \quad (6)$$

where  $m$  and  $M$  are complex parameters which are not fixed by the symmetry. The above  $M_\nu$  is diagonalized by  $U_\nu = V_\nu R_{12}(\pi/4)Q$ , where  $R_{12}(\pi/4)$  represents a maximal rotation in 1-2 plane and  $Q = \text{Diag.}(i, 1, 1)$ . The diagonal  $M_\nu$  has two degenerate eigenstates of mass  $|m|$  to be identified as the solar pair and third eigenstate with mass  $|M|$ . Two limits  $|m| \ll |M|$  and  $|m| \gg |M|$  correspond to the normal and the inverted ordering in neutrino masses respectively. A special case of the latter with  $|M| = 0$  corresponds to an enlarged residual symmetry which can be written as

$$S_{2\nu} \equiv V_\nu \text{Diag.}(\eta, \eta^*, \beta) V_\nu^\dagger \quad (7)$$

with  $\beta^k = 1$  and  $k \geq 3$ .  $M_\nu$  invariant under this symmetry will describe a massless plus a Dirac state. In both these cases described above, the lepton mixing matrix is given as

$$U_{\text{PMNS}} \equiv U = P_l (V_l^\dagger V_\nu R_{12}(\pi/4)Q) R_{12}(\theta_x) P_{\beta_2} \equiv P_l U^0 R_{12}(\theta_x) P_{\beta_2}, \quad (8)$$

where  $P_l$  is a diagonal phase matrix with arbitrary phases and  $V_l$  is defined in Eq. (2).  $R_{12}(\theta_x)$  represents an arbitrary rotation in 1-2 plane allowed due to the degeneracy in neutrino masses and  $P_{\beta_2} = \text{Diag.}(1, 1, e^{i\beta_2/2})$  is a Majorana phase matrix associated with only third neutrino. Both  $\theta_x$  and  $P_{\beta_2}$  cannot be fixed by the symmetries  $S_{1\nu}$  or  $S_{2\nu}$  and  $T_l$ . However, the matrix  $U^0$  is completely determined, up to the freedom in interchanging the rows and the first two columns, once  $S_{1\nu}$  or  $S_{2\nu}$  and  $T_l$  are known. In the following, we shall assume that  $S_{1\nu}$  or  $S_{2\nu}$  along with  $T_l$  represent the residual symmetries of a neutrino and the charged lepton mass matrices and look for a symmetry groups  $G_f$  which contain them. The choice of  $G_f$  and  $S_{1\nu}$ ,  $T_l$  within it will determine  $V_\nu$ ,  $V_l$  and hence  $U^0$  in Eq. (8). A  $T_l$  can be chosen such that  $\det(T_l) = 1$  and  $G_f$  can be a DSG of  $SU(3)$  when  $|m|, |M| \neq 0$ . In contrast, the group embedding  $S_{2\nu}$ ,  $T_l$  will be a DSG of  $U(3)$  when  $|M| = 0$ . We shall discuss these two cases separately in section III and V respectively.

At this point, it is important to identify the physical observables in  $U_{\text{PMNS}}$  which do not depend on the arbitrariness present in Eq. (8) and can be predicted at zeroth order from the underlying groups only. The following combinations [26] of the elements of  $U_{\text{PMNS}}$  are independent of the arbitrary parameters  $\theta_x$  and  $\beta_2$ :

$$\begin{aligned} |U_{\alpha 3}| &= |U_{\alpha 3}^0|, \\ \mathcal{I}_\alpha &\equiv \text{Im}(U_{\alpha 1}^* U_{\alpha 2}) = \text{Im}(U_{\alpha 1}^{0*} U_{\alpha 2}^0), \end{aligned} \quad (9)$$

where  $\alpha = e, \mu, \tau$  and repeated indices do not mean summation. Not all the three combinations in the second equation are independent as  $\mathcal{I}_e + \mathcal{I}_\mu + \mathcal{I}_\tau = 0$  follows from the unitarity of  $U$ . The first of Eq. (9) determines the mixing angles  $\theta_{23}$  and  $\theta_{13}$  without any ambiguity. The other two equations determine the specific combinations of the solar angle  $\theta_{12}$ , Dirac CP phase  $\delta$  and a Majorana phase  $\beta_1$ . These combinations in the standard parametrization are given as

$$\begin{aligned} c_{12}s_{12} \sin \frac{\beta_1}{2} &= \frac{1}{c_{13}^2} \mathcal{I}_e, \\ c_{12}^2 \sin \left( \delta - \frac{\beta_1}{2} \right) + s_{12}^2 \sin \left( \delta + \frac{\beta_1}{2} \right) &= \frac{1}{s_{23}c_{23}s_{13}} \left( \mathcal{I}_\mu - \frac{(s_{23}^2 s_{13}^2 - c_{23}^2)}{c_{13}^2} \mathcal{I}_e \right), \end{aligned} \quad (10)$$

where  $s_{ij} = \sin \theta_{ij}$  and  $c_{ij} = \cos \theta_{ij}$ . The quantities in the right side of the above equations can be determined from the underlying symmetries. Note that  $\theta_{12}$ ,  $\delta$  and  $\beta_1$  are unphysical if the solar neutrino pair is exactly degenerate. The small perturbations required to lift the degeneracy then determines the values of these observables. However, the above combinations between these observables are fixed at the leading order solely by the symmetries. Thus they are expected to receive small corrections just like  $\theta_{23}$  and  $\theta_{13}$ . Considering this, we provide the leading order predictions not only for  $\theta_{23}$  and  $\theta_{13}$  but also for the correlations given in Eqs. (10) in the following analysis.

### III. DSG OF $SU(3)$ AND NEUTRINO MIXING

All the DSG of  $SU(3)$  have been systematically classified in terms of their generators [27–32]. They are listed and further studied in [33–39]. In the following, we shall consider all the DSG of  $SU(3)$  having three dimensional IR and look for viability of these groups as a flavour symmetry for a Dirac neutrino and work out the predictions of the mixing angles. Specifically we shall look for the minimal DSG of  $SU(3)$  containing groups generated by  $S_{1\nu}$  and  $T_l$  of Eqs. (5) and (2) as subgroups. The  $T_l$  will be assumed to have determinant +1 and all three different eigenvalues. Since all eigenvalues of  $S_{1\nu}$  are also different, both  $V_\nu$  and  $V_l$  get uniquely determined apart from a diagonal phase matrix. Then using Eqs. (3) and (9) one can determine leading order predictions for  $\theta_{13}$  and  $\theta_{23}$  and the correlations listed in Eq. (10). Small perturbations generating the solar scale is not expected to change these predictions in a big way and we shall work out these predictions analytically and numerically for various groups. All the DSG of  $SU(3)$  having three dimensional IR are classified in three main categories: type C, type D and some other that do not fall in either of these two categories. In this section, we present a complete analytical study of the first two categories of the groups for their viability as a symmetry of degenerate solar pair together with realistic mixing pattern. A numerical analysis for the groups in all three categories will be presented in the next section.

#### A. $SU(3)$ subgroups of type C: The group series $C(n, a, b)$

The groups of type C, also characterized by a series  $C(n, a, b)$ , are isomorphic to groups generated from the following  $3 \times 3$  matrices [35, 37]:

$$F(n, a, b) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (11)$$

where  $\eta = e^{2\pi i/n}$  and  $a, b = 0, 1, 2, \dots, n-1$ . We shall first discuss a member  $\Delta(3n^2) \equiv C(n, 0, 1)$  of this series and then generalize the consideration to all the groups in the series.  $\Delta(3n^2)$  is isomorphic to  $(Z_n \times Z_n) \rtimes Z_3$ . Detailed properties of these group are studied in [9, 31]. All of its IR are either three or one dimensional.  $E$  and diagonal matrix  $F(n, 0, 1) = \text{Diag.}(1, \eta, \eta^*)$  provide generators of a faithful 3-dimensional representation. Their multiple products therefore generate the entire group whose elements can be labeled as:

$$W(n, a, b) \equiv W = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, \quad R(n, a, b) \equiv R = \begin{pmatrix} 0 & 0 & \eta^a \\ \eta^b & 0 & 0 \\ 0 & \eta^{-a-b} & 0 \end{pmatrix},$$

$$V(n, a, b) \equiv V = \begin{pmatrix} 0 & \eta^a & 0 \\ 0 & 0 & \eta^b \\ \eta^{-a-b} & 0 & 0 \end{pmatrix}. \quad (12)$$

Here  $a, b = 0, 1, 2, \dots, n-1$  and hence each of the non-zero entries of above three matrices take  $n$  different values corresponding to  $n^{\text{th}}$  roots of unity. Since one of the row in each of the matrices is determined from the other two due to their unit determinant, we get total  $3n^2$  independent elements of the group. In the following, we shall first assume that leptonic doublets are assigned to the 3-dimensional representation given in Eq. (12) and show that basic conclusion remains unchanged even if they are assigned to any other 3-dimensional representations.

In order to obtain a Dirac state through  $\Delta(3n^2)$ , one needs to find appropriate element of the group which can be used as  $S_{1\nu}$  in Eq. (5) and  $T_l$  in Eq. (2). There exists numerous choices for these. To see this, we note that all the non-diagonal elements  $R$  and  $V$  of the group posses identical set of eigenvalues  $(1, \omega, \omega^2)$  with  $\omega = e^{2\pi i/3}$ . Using this, we can easily enumerate all possible choices of the residual symmetries.  $S_{1\nu}$  can either be (a) a diagonal element in set  $W$  such that one of the non-zero entry is 1 and others are complex or (b) any of the matrices contained in  $R$  or  $V$ . Likewise,  $T_l$  can be (1) any element in  $W$  with unequal eigenvalues or (2) any element in  $R$  or  $V$ . Choices of  $S_{1\nu}$ ,  $T_l$  automatically determine the diagonalizing matrices  $V_\nu$ ,  $V_l$  and hence  $U^0$  in Eq. (8). Non-trivial mixing occurs only when at least one of the residual symmetries is chosen to be either  $R$  or  $V$ . Explicitly,  $R$  is diagonalized by:

$$V_R(n, a, b) = \text{Diag.}(1, \eta^b, \eta^{-a}) U_\omega \quad (13)$$

with

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (14)$$

Similarly,  $V$  is diagonalized by  $V_V(n, a, b) \equiv \text{Diag.}(1, \eta^{-a}, \eta^{-a-b}) U_\omega^*$ .

Combinations of possibilities for the choice of residual symmetries listed above determine all the mixing patterns that can be generated using the  $\Delta(3n^2)$  groups. As already mentioned, the choice (a) and (1) leads to  $V_\nu = V_l = I$  and vanishing  $\theta_{13}$  and  $\theta_{23}$  in Eq. (8). The choices, (a) and (2) or (b) and (1) contains either  $V_\nu$  or  $V_l$  proportional to identity while the other is given by  $V_R(n, a, b)$  or  $V_V(n, a, b)$ . It follows then from Eq. (8) that  $|U_{\alpha 3}|$  has a democratic structure and implies  $\sin^2 \theta_{13} = \frac{1}{3}$  which is very large. One therefore is left to consider the case (b) and (2). There are four possible structures for  $|U_{\text{PMNS}}|$  in this case but all give equivalent mixing and we explicitly consider the choice:

$$S_{1\nu} = \begin{pmatrix} 0 & 0 & \eta^{a_\nu} \\ \eta^{b_\nu} & 0 & 0 \\ 0 & \eta^{-a_\nu-b_\nu} & 0 \end{pmatrix}, \quad T_l = \begin{pmatrix} 0 & 0 & \eta^{a_l} \\ \eta^{b_l} & 0 & 0 \\ 0 & \eta^{-a_l-b_l} & 0 \end{pmatrix}, \quad (15)$$

The matrices diagonalizing above are respectively given by  $V_\nu = V_R(n, a_\nu, b_\nu) P_{13}$  and  $V_l = V_R(n, a_l, b_l)$ . The columns of the matrix  $V_R(n, a, b)$  given in Eq. (13) correspond to eigenvectors with eigenvalues  $(1, \omega, \omega^2)$ . If  $S_{1\nu}$  is to define a Dirac pair with first two degenerate eigenvalues then the first two columns of the diagonalizing matrix should correspond to the

eigenvalues which are complex conjugate to each other. This is done by inserting a matrix  $P_{13}$  which interchanges the first and the third column of  $V_R(n, a_\nu, b_\nu)$  to give proper  $V_\nu$ .

$V_\nu$  and  $V_l$  defined above determine the third column  $|U_{\alpha 3}|$  of the mixing matrix uniquely, see Eq. (9). Explicitly,

$$|U_{\alpha 3}|^2 = \frac{1}{9} (3 + 2 \cos x_\alpha + 2 \cos y_\alpha + 2 \cos(x_\alpha + y_\alpha)) . \quad (16)$$

Here  $\alpha = e, \mu, \tau$ , and

$$x_\alpha = \frac{2\pi}{n} \left( b_\nu - b_l + \frac{n}{3} p_\alpha \right), \quad y_\alpha = \frac{2\pi}{n} \left( a_\nu - a_l + \frac{n}{3} p_\alpha \right)$$

with  $p_\alpha = (0, 1, 2)$ . The entries  $|U_{\alpha 3}|^2$  can be interchanged by reordering the eigenvalues of  $T_l$ . Using this freedom, we define

$$\sin^2 \theta_{13} = \text{Min}(|U_{\alpha 3}|^2), \quad \sin^2 \theta_{23} \text{ or } \cos^2 \theta_{23} = \text{Max}(|U_{\alpha 3}|^2) . \quad (17)$$

Note that  $a_\nu, a_l, b_\nu, b_l$  vary over all possible  $n^{\text{th}}$  roots of unity corresponding to different choices of  $S_{1\nu}$  and  $T_l$  among the elements of  $\Delta(3n^2)$  and varying them over all roots exhaust all predictions within the group. It is easy to show that there does not exist any values of these parameters which can simultaneously give correct  $\theta_{13}$  and  $\theta_{23}$ . We demonstrate this graphically. Let us treat  $x_\alpha$  and  $y_\alpha$  as continuous variables and vary them in the entire range  $0-2\pi$ . The allowed discrete choices for  $a_{\nu,l}, b_{\nu,l}$  and  $n$  will clearly span a subset of this continuous range. Following this way and using Eqs.(17), the predictions for  $\theta_{13}$  and  $\theta_{23}$  are depicted in Fig. 1. All the possible predictions for  $\theta_{13}$  and  $\theta_{23}$  for the group series  $\Delta(3n^2)$  lie as discrete points in the red shaded region. As can be seen from figure,  $\Delta(3n^2)$  groups can at most give good leading order prediction either for  $\theta_{13}$  or  $\theta_{23}$  but not for both. If one is near to the  $3\sigma$  limit then the other is far away from it. Thus these groups are not suitable for the descriptions of a Dirac neutrino with correct leading order predictions of mixing angles.

In deriving the above conclusion, we had used the fact that three generations of leptonic doublets transform as a triplet representation generated using  $E$  and  $F(n, 0, 1)$ . The group admits other 3-dimensional IR labeled by  $3_{(k,l)}$  [31]. These are generated using  $E, F(n, k, l)$  given in Eq. (11) with the provision that values of  $(k, l)$  giving equivalent representation should be excluded [31]. The forgoing argument gets carried over even when leptons are assigned to any general triplet IR  $3_{(k,l)}$  and not just  $3_{(0,1)}$  since it only used the basic texture for elements of the  $\Delta(3n^2)$  groups given in Eq. (12). The same texture gets carried over for all the triplet representation except that non-zero entries in these elements now need not span all the  $n^{\text{th}}$  roots. Thus the leading order prediction will only be a subset of the ones predicted using the representation  $3_{(0,1)}$ . Since the latter fails in simultaneously giving correct mixing angles, assigning leptons to a different triplet will not alter the general conclusion.

The argument in the forgoing para can be used not just to rule out other IR of  $\Delta(3n^2)$  but the entire group series  $C(n, a, b)$ . This follows from the observation [37] that the generators of the series  $C(n, a, b)$  as given in Eq. (11) also represent 3-dimensional IR of the groups



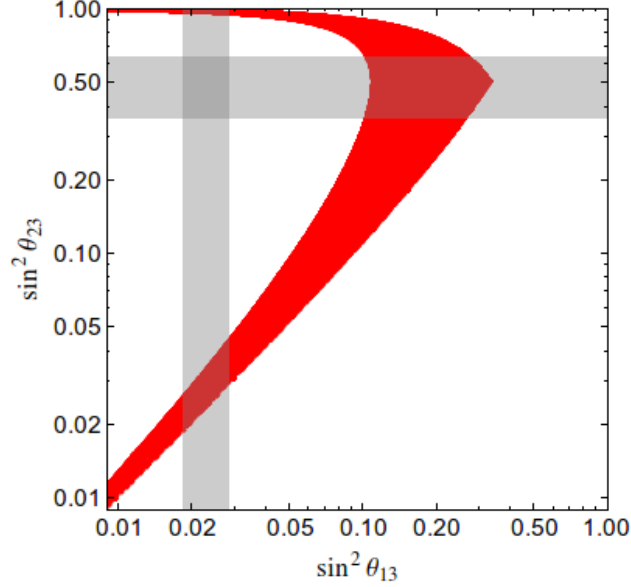


FIG. 1. Predictions for  $\sin^2 \theta_{13}$  and  $\sin^2 \theta_{23}$  obtained in case of the C-type DSG of  $SU(3)$ . The allowed values lie at some discrete points in the red shaded region. The horizontal and vertical gray bands correspond to the respective  $3\sigma$  ranges from the global fits [40].

$\Delta(3n^2)$ . These groups are isomorphic to matrix groups generated using the IR of  $\Delta(3n^2)$  and thus any of the groups  $C(n, a, b)$  cannot lead to correct  $\theta_{13}$  and  $\theta_{23}$  simultaneously. This negative result is useful in discarding large number of small DSG of  $SU(3)$  from our consideration as a possible candidate of symmetry in the lepton sector. For example, among the first 59 DSG of  $SU(3)$  of order  $< 512$  listed in [37], 46 groups fall in this category and can be ruled out using the above results.

### B. $SU(3)$ subgroups of type D: The group series $D(n, a, b; d, r, s)$

The groups characterized by a series  $D(n, a, b; d, r, s)$  are the groups obtained by adding the following matrices to the generators  $E$  and  $F(n, a, b)$  of the series  $C(n, a, b)$  [32, 35]:

$$G(r, d, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix}, \quad (18)$$

where  $\delta = e^{2\pi i/d}$  with integer  $d$  and  $r, s = 0, 1, \dots, d-1$ . We shall be using the following important results in our discussion of these groups:

- (1)  $\Delta(6n^2) \equiv D(n, 0, 1; 2, 1, 1)$  provides one of the infinite series within the type D groups.
- (2) Every  $D(n, a, b; d, r, s)$  is a subgroup of the group  $\Delta(6g^2)$  with  $g = \text{lcm}(n, d, 2)$  [38].

- (3) All the D-type groups are contained in only one of the three infinite series of groups [35], namely  $\Delta(6n^2)$ ,  $Z_3 \times \Delta(6n^2)$  and  $D_{9n,3n}^1$  which is isomorphic to  $(Z_{9n} \times Z_{3n}) \rtimes S_3$ .

Let us first consider the  $\Delta(6n^2) = D(n, 0, 1; 2, 1, 1)$  groups whose finding can be used to make some general statement on the other groups in the series. Earlier study of these groups in the context of three non-degenerate neutrinos was presented in [41]. Addition of the generator  $G(2, 1, 1)$  lead to  $3n^2$  new elements apart from the  $3n^2$  elements, Eq. (12), of  $\Delta(3n^2)$ :

$$\begin{aligned} S \equiv S(n, a, b) &= - \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & 0 & \eta^b \\ 0 & \eta^{-a-b} & 0 \end{pmatrix}, \quad T \equiv T(n, a, b) = - \begin{pmatrix} 0 & 0 & \eta^b \\ 0 & \eta^a & 0 \\ \eta^{-a-b} & 0 & 0 \end{pmatrix}, \\ U \equiv U(n, a, b) &= - \begin{pmatrix} 0 & \eta^b & 0 \\ \eta^{-a-b} & 0 & 0 \\ 0 & 0 & \eta^a \end{pmatrix}. \end{aligned} \quad (19)$$

Here also  $a, b = 0, 1, \dots, n-1$ . These new elements give more freedom in choosing residual symmetries compared to  $\Delta(3n^2)$ . To enumerate these choices, let us note that eigenvalues of  $S, T, U$  are given by  $(-\eta^a, \eta^{-\frac{a}{2}}, -\eta^{-\frac{a}{2}})$ . The corresponding diagonalizing matrices are:

$$\begin{aligned} V_S(a, b) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & \eta^{b+a/2} \\ 0 & -\eta^{-b-a/2} & 1 \end{pmatrix}, \quad V_T(a, b) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \eta^{b+a/2} \\ 0 & \sqrt{2} & 0 \\ -\eta^{-b-a/2} & 0 & 1 \end{pmatrix}, \\ V_U(a, b) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \eta^{b+a/2} & 0 \\ -\eta^{-b-a/2} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \end{aligned} \quad (20)$$

The  $S, T, U$  contain two complex conjugate eigenvalues only for even  $n$  and when  $a = n/2$ . The corresponding eigenvalues are  $(i, -i, 1)$ . Except in this special case, the above  $S, T, U$  cannot serve as the desired residual symmetry of neutrino mass matrix. It follows from the structures of the diagonalizing matrices and Eq. (8) that if  $S_{1\nu}$  or  $T_l$  is diagonal, *i.e.*  $\sim W$ , then the third column of  $U_{\text{PMNS}}$  is either  $1/\sqrt{3}(1, 1, 1)^T$  or  $1/\sqrt{2}(0, 1, 1)^T$  or its permutations leading to very large or vanishing  $\theta_{13}$ . The latter result is similar to the prediction of the  $\mu$ - $\tau$  symmetry. Thus both  $S_{1\nu}$  and  $T_l$  must be non-diagonal in order to obtain non-trivial prediction. Thus, we are left with four cases:

- (i) Both  $S_{1\nu}, T_l \in \{R, V\}$
- (ii)  $T_l \in \{R, V\}$  and  $S_{1\nu} \in \{S, T, U\}$  with eigenvalues  $(i, -i, 1)$ .
- (iii) Both  $S_{1\nu}, T_l \in \{S, T, U\}$  and  $S_{1\nu}$  has eigenvalues  $(i, -i, 1)$
- (iv)  $S_{1\nu} \in \{R, V\}$  and  $T_l \in \{S, T, U\}$

The case (i) has already been discussed and leads to the results shown in Fig. 1. The cases (ii) and (iii) respectively fix the third column of  $U_{\text{PMNS}}$  to  $1/\sqrt{3}(1, 1, 1)^T$  and  $1/\sqrt{2}(0, 1, 1)^T$  or its permutations. The case (iv) can give non-zero and small  $\theta_{13}$  and we consider it explicitly. Let us choose  $S_{1\nu} = R(n, a_\nu, b_\nu)$  and  $T_l = T(n, a_l, b_l)$ . Other choices within this set do not give any new results. The corresponding diagonalizing matrices are given by  $V_\nu = V_R(n, a_\nu, b_\nu)P_{13}$  and  $V_l = V_T(n, a_l, b_l)$  with  $V_R$  and  $V_T$  as in Eqs. (13) and (20) respectively. As before, a permutation matrix  $P_{13}$  is introduced to ensure that the eigenvectors are ordered according to the eigenvalues  $(\omega^2, \omega, 1)$ . The third column of the mixing matrix then follows from Eq. (8) and leads to the following predictions:

$$\sin^2 \theta_{13} = \frac{2}{3} \text{Min} \left( \cos^2 \left( \pi \frac{b_l - a_\nu + \frac{a_l}{2}}{n} \right), \sin^2 \left( \pi \frac{b_l - a_\nu + \frac{a_l}{2}}{n} \right) \right),$$

$$(\sin^2 \theta_{23} \text{ or } \cos^2 \theta_{23}) = \frac{1}{3 \cos^2 \theta_{13}}. \quad (21)$$

One could rearrange the entries in the third column by changing the order of the eigenvalues of  $T_l$ . This corresponds to interchange  $\sin^2 \theta_{23} \leftrightarrow \cos^2 \theta_{23}$  and one gets two possibilities given in the last equation above. Note that the second in Eq. (21) is universal and is dictated by the textures of the chosen  $S_{1\nu}$ ,  $T_l$  and not by the actual values of the parameters  $a_\nu, b_l, a_l$ . This means that this prediction can be obtained in all the groups of the series  $\Delta(6n^2)$  by choosing  $S_{1\nu}$  and  $T_l$  as in this case. One may regard this universal prediction as a very good zeroth order choice. Inserting the best fit value  $\sin^2 \theta_{13} = 0.023$  in Eq. (21) gives  $\sin^2 \theta_{23} = 0.341$  or  $0.658$  which is though outside the allowed  $3\sigma$  range is quite close to it and small perturbation may bring it within the required range. The other invariant quantities expressed in Eq. (9) can also be determined for the above choice of  $S_{1\nu}$  and  $T_l$  as

$$\begin{aligned} \mathcal{I}_\mu = 0, \mathcal{I}_e = -\mathcal{I}_\tau &= \pm \frac{1}{2\sqrt{3}} \sin \left( 2\pi \frac{b_l - a_\nu + \frac{a_l}{2}}{n} \right) \\ &= \pm \frac{\sqrt{3}}{2} \sin \theta_{13} \cos \theta_{13} \cos \theta_{23} \quad \text{for } \sin^2 \theta_{23} \cos^2 \theta_{13} = \frac{1}{3}, \\ \mathcal{I}_\tau = 0, \mathcal{I}_e = -\mathcal{I}_\mu &= \pm \frac{1}{2\sqrt{3}} \sin \left( 2\pi \frac{b_l - a_\nu + \frac{a_l}{2}}{n} \right) \\ &= \pm \frac{\sqrt{3}}{2} \sin \theta_{13} \cos \theta_{13} \sin \theta_{23} \quad \text{for } \cos^2 \theta_{23} \cos^2 \theta_{13} = \frac{1}{3}, \end{aligned} \quad (22)$$

where we have used Eq. (21) in writing the above equation. Note that at least one of the  $\mathcal{I}_\alpha$ s is vanishing for all the groups of the series  $\Delta(6n^2)$ . In addition to the above, there are two other universal predictions possible within the groups in the  $\Delta(6n^2)$  series. First one is obtained when  $a_l, b_l, a_\nu$  are chosen as zero which is always an allowed choice. From Eq. (21), one gets in this case  $\theta_{13} = 0$  and  $\sin^2 \theta_{23} = \frac{1}{3}$ . The other prediction corresponds to the case (iii) discussed above, *i.e.*  $\theta_{13} = 0$  and  $\theta_{23} = \frac{\pi}{4}$ , and follows in all the  $\Delta(6n^2)$  groups with even  $n$ .

There exists several choices for  $n$  and  $a_\nu, b_\nu, a_l, b_l$  which can give correct  $\theta_{13}$  in the  $3\sigma$  range allowed by the global fits [40]. These are group specific. We explore this numerically

for first few values of  $n \leq 50$ . The results are displayed in Fig. 2. It is seen from the figure that the smallest such value occurs for  $n = 8$  which corresponds to an order 324 group  $\Delta(6 \cdot 8^2)$ . Further, for the values of  $8 < n \leq 50$ , all  $n$  except  $10 \leq n \leq 14$  and  $n = 20, 21$  can lead to the value of  $\theta_{13}$  preferred by global fits at  $3\sigma$ .

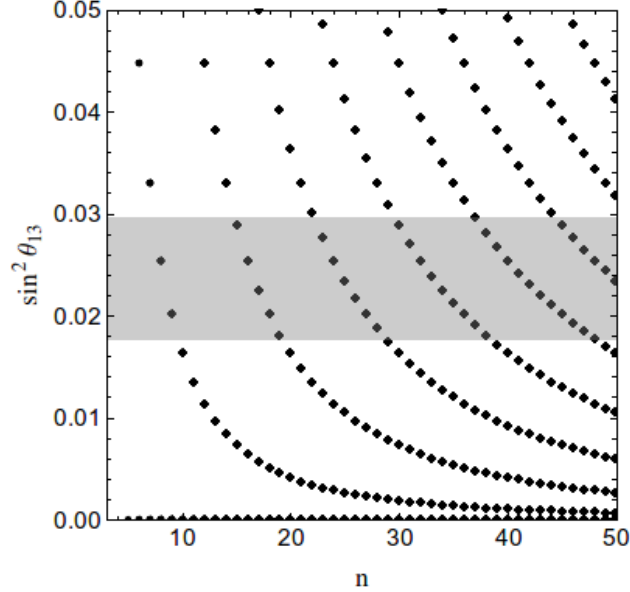


FIG. 2. Predictions for  $\sin^2 \theta_{13}$  as function of  $n$  for  $n \leq 50$  in case of the  $\Delta(6n^2)$  groups. The horizontal gray band corresponds to the  $3\sigma$  region in  $\sin^2 \theta_{13}$  allowed by the global fits [40].

The above considerations can be easily generalized to all the DSG of  $SU(3)$  of type D since latter are always subgroups of some  $\Delta(6g^2)$  as already mentioned. As a consequence, all the elements of any D-type groups have the same six possible textures as in the case of  $\Delta(6n^2)$ , Eqs. (12) and (19). In particular, choices of  $S_{1\nu}$  and  $T_l$  as discussed in case (iv) for  $\Delta(6n^2)$  always exist for any type D groups and the resulting prediction, Eq. (21) always holds but allowed values of parameters  $a_l, b_l, a_\nu$  may now form a subset of the possible values in case of  $\Delta(6g^2)$ . In particular, the universal prediction, second of Eqs. (21) which is independent of the values of these parameters will always follow in all the groups of type D. Allowed values of  $\theta_{13}$  will be a subset of the prediction in Fig 2. We shall explore this numerically for the group series  $D_{9n,3n}^1$  in the next subsection along with the other groups.

#### IV. NUMERICAL STUDY OF DSG OF $SU(3)$

We now numerically study various DSG of  $SU(3)$ . This study includes (1) all DSG of  $SU(3)$  with order  $< 512$  as tabulated by Ludl in [37], (2) the remaining groups  $A_5, \Sigma(108), \Sigma(168), \Sigma(216), \Sigma(648), \Sigma(1060)$  which are not of type C or D and (3) some of the larger groups of type D labeled as  $D_{m,n}^1$  by Grimus and Ludl [35]. Numerical procedure followed is the following. We first construct all the elements of a group falling in the three categories

described above from the defining generators. We then identify all possible elements which can be chosen as possible residual symmetries  $S_{1\nu}$  and  $T_l$  of the neutrino and the charged leptons respectively such that  $S_{1\nu}$  and  $T_l$  do not commute and construct all possible mixing matrices from their diagonalization. The results are displayed in Table I in which we collect

Group	GAP code	$\sin^2 \theta_{13}$	$\sin^2 \theta_{23}$	$\mathcal{I}_e = \text{Im}(U_{e1}^* U_{e2})$	$\mathcal{I}_\mu = \text{Im}(U_{\mu 1}^* U_{\mu 2})$
$\Delta(6 \cdot 6^2)$	[[216,95]]	0.045	0.349	$\pm 0.144$	0
$\Delta(6 \cdot 7^2)$	[[294,7]]	0.033	0.345	$\pm 0.125$	0
$\Delta(6 \cdot 8^2)$	[[384,568]]	0.025★	0.342	$\pm 0.11$	0
$\Delta(6 \cdot 9^2)$	[[486,61]]	0.02★	0.34	$\pm 0.099$	0
$D_{9,3}^1 \cong (Z_9 \times Z_3) \rtimes S_3$	[[162,14]]	0.02★	0.34	$\pm 0.099$	0
$D_{18,6}^1 \cong (Z_{18} \times Z_6) \rtimes S_3$	[[648,259]]	0.005	0.335	$\pm 0.05$	0
		0.02★	0.34	$\pm 0.099$	0
		0.045	0.349	$\pm 0.144$	0
$D_{27,9}^1 \cong (Z_{27} \times Z_9) \rtimes S_3$	[[1458,658]]	0.002	0.334	$\pm 0.034$	0
		0.009	0.336	$\pm 0.067$	0
		0.02★	0.34	$\pm 0.099$	0
		0.035	0.346	$\pm 0.13$	0
$A_5 \cong \Sigma(60)$	[[60,5]]	0.035	0.5★	0	$\pm 0.094$
$PSL(2, 7) \cong \Sigma(168)$	[[168,42]]	0.023★	0.455★	$\pm 0.377$	$\mp 0.255$
		0.027★	0.293	$\pm 0.333$	$\mp 0.185$
		0.044	0.262	$\pm 0.353$	$\mp 0.331$
$\Sigma(648)$	[[648,532]]	0.012	0.279	$\pm 0.342$	$\mp 0.284$
		0.019★	0.341	$\pm 0.406$	$\mp 0.307$
		0.02★	0.34	$\pm 0.099$	0
$\Sigma(1080)$	[[1080,260]]	0.006	0.364★	$\pm 0.392$	$\mp 0.228$
		0.016	0.226	$\pm 0.27$	$\mp 0.167$
		0.035	0.458★	$\pm 0.283$	$\mp 0.079$
		0.035	0.5★	0	$\pm 0.094$
		0.039	0.319	$\pm 0.153$	$\mp 0.019$
		0.042	0.392★	$\pm 0.436$	$\mp 0.27$
		0.045	0.349	$\pm 0.144$	0

TABLE I. Predictions for  $\theta_{23}$ ,  $\theta_{13}$  and  $\mathcal{I}_\alpha$  with two degenerate neutrinos and third massive neutrino from a scan of various DSG of  $SU(3)$ . The prediction for  $\sin^2 \theta_{23}$  is chosen to be in the first octant. Only those groups which give  $0 < \sin^2 \theta_{13} \leq 0.05$  and  $0.2 \leq \sin^2 \theta_{23} \leq 0.5$  are included. The predictions marked by ★ (\*) are within the  $1\sigma$  ( $3\sigma$ ) interval of the global fits [40]. The group  $[[g, j]]$  refers to the  $j^{\text{th}}$  finite group of order  $g$  as classified in the Small Group Library in GAP [42].

only the solutions which give  $0 < \sin^2 \theta_{13} \leq 0.05$  and  $0.2 < \sin^2 \theta_{23} \leq 0.5$ . We have chosen

$\theta_{23}$  in the first octant and its other predicted value can be obtained by reading fourth column in Table I as  $\cos^2 \theta_{23}$ . We also give the predictions for  $\mathcal{I}_e$  and  $\mathcal{I}_\mu$  which determine certain correlations between the solar angle, Dirac CP phase and one of the Majorana phases as displayed in Eq. (10). Numerical results are consistent with the analytic results discussed in the last section. Noteworthy features of the table are:

- None of the C-type groups feature in the table as they cannot give correct  $\theta_{13}$  and  $\theta_{23}$  simultaneously.
- The D-type groups shown in table are first few successful members of the series  $D_{n,n}^0 = \Delta(6n^2)$  and  $D_{9n,3n}^1$ . In all the cases, the atmospheric mixing angle determined numerically satisfies the second of Eq. (21). Universal prediction  $\theta_{13} = 0$  and  $\sin^2 \theta_{23} = \frac{1}{3}$  are found to be true for all the D-type groups for  $n \geq 3$  and we do not show them in the Table. Likewise  $\Delta(6n^2)$  with even  $n$  and  $n \geq 4$  also give the  $\mu$ - $\tau$  symmetric prediction. These cases will need significant non-leading corrections to explain the experimental value. The last two are the only possible prediction for  $n \leq 5$ . Because of this, the familiar small groups in type D like  $S_4 \cong \Delta(24)$ ,  $\Delta(96)$  do not appear in Table I. Various groups with  $n \geq 8$  can predict  $\theta_{13}$  within the allowed  $3\sigma$  range.
- We also show prediction obtained in case of three other groups namely  $D_{9,3}^1$ ,  $D_{18,6}^1$  and  $D_{27,9}^1$  with order 162, 648 and 1458 respectively. They do not lead to any new useful leading order prediction for  $\theta_{13}$ .
- The prediction for the atmospheric mixing angle in case of the non D-type groups are better. In particular, the group  $\Sigma(168) \cong PSL(2, 7)$  predicts  $\theta_{13}$  very close to the best fit value and  $\theta_{23}$  within  $1\sigma$  of the best fit value. The mixing angle predictions for this group in the case of three non degenerate neutrinos were presented in [16]. Our predictions are different due to difference in the chosen residual symmetry.
- The  $A_5 \cong \Sigma(60)$  is the only group in the Table I which belongs to the category of finite von-Dyck groups [17, 18] investigated earlier as a symmetry of degenerate solar pair in [26]. Our results of this group are in agreement with their findings. In particular, this group predicts vanishing  $\beta_1$  and maximal  $\delta$  from Eq. (10) and Table I if solar angle is non-zero as would be the case with any realistic perturbation. We shall explore this prediction further in section VI.

## V. FLAVOUR SYMMETRIES FOR DEGENERATE NEUTRINO PAIR AND A MASSLESS NEUTRINO

We now discuss a different class of symmetries which provide a good zeroth order spectrum for an inverted hierarchy, *i.e.* two degenerate neutrinos ( $\nu_1, \nu_2$ ) which are mix of flavour states and a third massless neutrino  $\nu_3$ .

Flavour symmetries for a massless neutrino are extensively discussed in [24, 25] in the case when other two neutrinos are non-degenerate. We now extend the same analysis to the case when massive neutrinos are degenerate. As discussed, this can be obtained if the residual symmetry of the neutrino masses is given by Eq. (7). Thus, we look for groups  $G_f$  which contain groups generated by  $S_{2\nu}$  and  $T_l$  as sub-groups.  $G_f$  has to be a subgroup of  $U(3)$  since  $\det(S_{2\nu}) \neq \pm 1$ . Unlike in the case of  $SU(3)$ , not all the DSG of  $U(3)$  are classified. But Ludl [37] has identified all the DSG of  $U(3)$  with order  $< 512$  and has also found several infinite series of groups (see also, [20]). In the following, we numerically investigate all the groups with order  $< 512$  having three dimensional IR as tabulated by Ludl. Before doing this, we discuss some general analytic properties useful in understanding numerical results to be derived.

We divide DSG of  $U(3)$  in two classes called X and Y in [25]. The groups in class X are generated by combination of six matrices having the same textures as  $R, S, T, U, V, W$  defined earlier in Eqs. (12,19). But now determinant of these generators is not required to be one and thus each of these are functions of four parameters  $\eta, a, b, c$  instead of three. Explicit forms of these are given by Ludl [37]. Given these forms, one argues [25] that all the elements of any group  $G_f$  in category X has six possible textures labeled as  $\tilde{R}, \tilde{S}, \tilde{T}, \tilde{U}, \tilde{V}, \tilde{W}$ . These are obtained respectively from  $R, S, T, U, V, W$  by replacing non-zero entries by some root of unity. For example,

$$\tilde{R}(\eta_1, \eta_2, \eta_3) = \begin{pmatrix} 0 & 0 & \eta_1 \\ \eta_2 & 0 & 0 \\ 0 & \eta_3 & 0 \end{pmatrix}, \quad \tilde{S}(\eta_1, \eta_2, \eta_3) = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & \eta_2 \\ 0 & \eta_3 & 0 \end{pmatrix}, \quad (23)$$

where  $\eta_{1,2,3}$  are some roots of unity. The eigenvalues of  $\tilde{A} \in \{\tilde{R}, \tilde{V}\}$  are given by  $\det(\tilde{A})^{\frac{1}{3}}(1, \omega, \omega^2)$  with  $\omega^3 = 1$ . Thus they do not contain a pair of eigenvalues which are complex conjugate to each other as long as  $\det(\tilde{A}) \neq 1$ . Such elements cannot therefore be chosen as a possible  $S_{2\nu}$ . The latter can come either from the diagonal structure  $\tilde{W}$  or non-diagonal structure  $\tilde{S}, \tilde{T}, \tilde{U}$  having only one non-zero diagonal entry say  $\eta_1$ . The other non-zero entries are labeled as  $\eta_2, \eta_3$ . The eigenvalues of matrices in this class are  $\eta_1, \pm\sqrt{\eta_2\eta_3}$  and they can have a complex pair of eigenvalues which can be used to obtain a massless state. The diagonal matrix  $\tilde{W}$  can also be chosen as either  $T_l$  or  $S_{2\nu}$  for appropriate values of its elements. But choosing either  $S_{2\nu}$  or  $T_l$  as diagonal does not give correct mixing pattern. Thus, one has essentially two non-trivial possibilities: (i)  $T_l \in \{\tilde{R}, \tilde{V}\}$  and  $S_{2\nu} \in \{\tilde{S}, \tilde{T}, \tilde{U}\}$  and (ii) both  $T_l$  and  $S_{2\nu} \in \{\tilde{S}, \tilde{T}, \tilde{U}\}$ .

Since scenario under consideration applies only to the inverted hierarchy, the flavour composition of the massless state determines the third column of the mixing matrix and hence  $\theta_{13}$  and  $\theta_{23}$ . Clearly, the case (ii) can only give  $\mu$ - $\tau$  symmetric structure as a non-trivial prediction. Let us thus consider the case (i) above and choose as an example,  $S_{2\nu} = \tilde{S}(\eta_{1\nu}, \eta_{2\nu}, \eta_{3\nu})$  and  $T_l = \tilde{R}(\eta_{1l}, \eta_{2l}, \eta_{3l})$  explicitly given in Eq. (23). Two eigenvalues of  $S_{2\nu}$

can be complex conjugate if<sup>1</sup>

$$\eta_{1\nu}^* = \pm(\eta_{2\nu}\eta_{3\nu})^{\frac{1}{2}}. \quad (24)$$

Eigenvector of  $S_{2\nu}$  corresponding to the massless state in this case is given by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(0, 1, \mp(\eta_{2\nu}^*\eta_{3\nu})^{\frac{1}{2}})^T$$

The  $U_l$  diagonalizing  $T_l$  is given by

$$U_l = V_{\bar{R}} = \text{Diag.}(1, p_l^* \eta_{2l}, p_l \eta_{1l}^*) U_\omega,$$

where  $U_\omega$  is defined in Eq. (14) and  $p_l = (\eta_{1l}\eta_{2l}\eta_{3l})^{\frac{1}{3}}$ . One thus gets flavour structure of massless state by operating  $U_l$  on  $|\psi_0\rangle$ . This leads to

$$|U_{\alpha 3}|^2 = \frac{1}{6} |p_l \eta_{2l}^* \lambda_\alpha \mp p_l^* \lambda_\alpha^* \eta_{1l} (\eta_{2\nu}^* \eta_{3\nu})^{\frac{1}{2}}|^2, \quad (25)$$

where  $\alpha = e, \mu, \tau$  and  $\lambda_\alpha = (1, \omega, \omega^2)$ .

This is to be compared with result derived in [25] where structure of massless state for all the groups in category X was determined. In particular, it was shown that only possible flavour probabilities for a massless state in case of the inverted hierarchy is  $(0, 1/2, 1/2)$  and its permutation. Unlike this, one can now find non-trivial  $\theta_{13}$  with the inverted hierarchy when the solar pair is degenerate at the leading order. The reason for this can be traced to different structure of  $S_{2\nu}$  which implies a different structure for  $|\psi_0\rangle$  compared to one with non-degenerate solar pair at the leading order. Eq. (25) leads to non-trivial prediction for

GAP code	Group	$\sin^2 \theta_{13}$	$\sin^2 \theta_{23}$	$\mathcal{I}_e = \text{Im}(U_{e1}^* U_{e2})$	$\mathcal{I}_\mu = \text{Im}(U_{\mu 1}^* U_{\mu 2})$
$\llbracket 96, 65 \rrbracket$	$S_4(3)$	0.045	0.349	$\pm 0.144$	0
$\llbracket 324, 13 \rrbracket$		0.02*	0.399★	$\pm 0.157$	$\mp 0.029$
$\llbracket 384, 571 \rrbracket$	$\Delta(6 \cdot 4^2, 3)$	0.045	0.349	$\pm 0.144$	0
$\llbracket 216, 25 \rrbracket$		0.033	0.388*	$\pm 0.108$	$\mp 0.154$
$\llbracket 432, 273 \rrbracket$		0.033	0.388*	$\pm 0.108$	$\mp 0.154$

TABLE II. Results obtained for DSG of  $U(3)$  of order  $< 512$  as listed in [37]. The other details are same as in the caption of Table I.

the third column which we now explore numerically for all the 75 groups tabulated as DSG of  $U(3)$  in [37]. Inspection of Table V in [37] coupled with analytic argument shows that only 22 of the 75 groups in category X can have element in class (i) and thus can give non-trivial prediction [25]. Along with these, five groups in category Y also have to be studied numerically. We generate all the elements of these 27 groups numerically. Then identify all

<sup>1</sup> The alternative possibility,  $\eta_1 \neq \pm 1, \eta_2 \eta_3 = -1$  gives a massless state with the  $\mu$ - $\tau$  symmetric structure as already found in [25].



the elements with a pair of complex conjugate eigenvalues and take them as possible  $S_{2\nu}$  and elements with three distinct eigenvalues are chosen as  $T_l$ . By diagonalizing these sets, we work out all possible mixing matrices and identify those which can give reasonable leading order prediction. It turns out that 20 of the 27 cases do not contain any element which can be identified as  $S_{2\nu}$ . Two of the remaining seven groups, namely  $[[162,10]]$  and  $[[486,125]]$ , give predictions similar to the one that follows from  $\mu$ - $\tau$  symmetry, *i.e.* vanishing  $\theta_{13}$  and maximal  $\theta_{23}$ . The other five groups give reasonably good leading order predictions for the mixing angles. These are shown in Table II. Noteworthy groups are  $[[324,13]]$  in category X and last two groups in the Table belonging to category Y. These lead to the atmospheric angle in the  $3\sigma$  range of the best fit value and  $\theta_{13}$  close to its  $3\sigma$  range. Thus it is conceivable that small perturbations in these two case can lead the desired solution.

## VI. GENERATING SOLAR SCALE: EXAMPLES OF PERTURBATIONS

Our discussion so far was restricted to the leading order predictions for  $\theta_{23}$  and  $\theta_{13}$  while  $\theta_{12}$  was undefined due to the exact degeneracy in solar pair. We now discuss possible perturbations which lift the degeneracy and hence generate the solar scale and the required value for the solar angle. We find that a common symmetry

$$S_1 \equiv E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (26)$$

and different  $T_l$  lead to the various predictions listed in Table I in case of  $\Delta(6n^2)$  and  $A_5$  groups. This is also applicable to the first two solutions obtained from the group  $PSL(2, 7)$ . Let us consider specific choices for  $T_l$  for these groups. The smallest  $\Delta(6n^2)$  leading to  $\sin^2 \theta_{13}$  within  $3\sigma$  is  $\Delta(384)$  in which case  $S_1$  and

$$T_l = U(8, 1, 0) = - \begin{pmatrix} 0 & 1 & 0 \\ \eta_8^* & 0 & 0 \\ 0 & 0 & \eta_8 \end{pmatrix} \quad (27)$$

with  $\eta_8 = e^{\frac{i\pi}{4}}$  leads to the prediction given in Table I. Analogous  $T_l$  in case of  $A_5$  and  $PSL(2, 7)$  are given by

$$T_l \equiv F(2, 0, 1)HE = \frac{1}{2} \begin{pmatrix} \mu_+ & -1 & \mu_- \\ 1 & -\mu_- & -\mu_+ \\ -\mu_- & -\mu_+ & 1 \end{pmatrix} \quad \text{for } A_5 \text{ and}$$

$$T_l \equiv NME = \frac{i}{\sqrt{7}} \begin{pmatrix} \beta^4(\beta - \beta^6) & \beta(\beta^4 - \beta^3) & \beta^2(\beta^2 - \beta^5) \\ \beta^4(\beta^4 - \beta^3) & \beta(\beta^2 - \beta^5) & \beta^2(\beta - \beta^6) \\ \beta^4(\beta^2 - \beta^5) & \beta(\beta - \beta^6) & \beta^2(\beta^4 - \beta^3) \end{pmatrix} \quad \text{for } PSL(2, L), \quad (28)$$

where  $\mu_{\pm} = \frac{1}{2}(-1 \pm \sqrt{5})$ ,  $\beta = e^{2\pi i/7}$  and  $H, N, M$  are generators given in [37].

The neutrino mass matrix invariant under  $S_1$  in Eq. (26) has the structure

$$M_\nu^0 = m_0 \begin{pmatrix} 1 & y & y \\ y & 1 & y \\ y & y & 1 \end{pmatrix}. \quad (29)$$

This form has been discussed as a leading order neutrino mass matrix in models based on the  $S_3$  symmetry [3]. Here the same form arises due to the fact that  $S_1$  is indeed an element of the group  $S_3$ . Important difference is that  $M_\nu^0$  above is not in the flavour basis since  $M_l M_l^\dagger$  invariant under  $T_l$  is not diagonal. Thus leading order predictions in two cases are entirely different.

The residual symmetry of neutrinos  $G_\nu$  characterized by  $S_1$  must be broken in order to generate the splitting in solar neutrinos and corrections to  $M_\nu^0$  may arise in various ways in the realistic models. If such corrections are small, the modified neutrino mass matrix in the presence of the most general perturbations that may arise from the breaking of  $G_\nu$  can suitably be expressed as

$$M_\nu = m_0 \begin{pmatrix} 1 + \epsilon_1 & y(1 + \epsilon_3) & y(1 + \epsilon_4) \\ y(1 + \epsilon_3) & 1 + \epsilon_2 & y \\ y(1 + \epsilon_4) & y & 1 \end{pmatrix}, \quad (30)$$

where  $\epsilon_i$  are in general complex parameters and  $|\epsilon_i| \ll 1$ . In the diagonal basis of the charged leptons, one obtains

$$M_\nu^f = V_l^T M_\nu V_l, \quad (31)$$

where  $V_l$  is the unitary matrix which diagonalizes  $M_l M_l^\dagger$  and is determined from the underlying group using Eq. (2) for specific  $T_l$  as given above as long as symmetry corresponding to  $T_l$  is unbroken. The corrected PMNS matrix can be obtained by diagonalizing the above  $M_\nu^f$  up to a freedom in interchanging the rows. The perturbations in  $M_\nu$  modifies the leading order predictions of mixing angles. We numerically study all three cases mentioned above as they lead to reasonably good predictions at the leading order. For this, we randomly vary all the parameters in  $M_\nu^f$  considering them to be complex and evaluate the neutrino masses and mixing angles for a given  $T_l$ . The overall undetermined scale  $m_0$  in Eq. (30) is fixed by requiring the correct atmospheric scale. The results are displayed in Fig. 3 for the group  $\Delta(6 \cdot 8^2)$ , in Fig. 4 for the group  $A_5$  and in Fig. 5 for the group  $PSL(2, 7)$ . All the points in the left panels in the Figs. 3, 4 and 5 are obtained for  $|\epsilon_i| < 0.05$  and requiring that they reproduce solar and atmospheric mass squared differences and the solar mixing angle in the  $3\sigma$  ranges of their global fit values [40]. Though the solar mixing angle is not predicted at the leading order by the symmetries, even small  $\epsilon_i$  can generate its required value because of the degeneracy in solar pair. It can be seen from all the three figures that the reactor angle is more sensitive to the small perturbations while the atmospheric mixing angle receives relatively small corrections. Hence such corrections are more suitable for the group  $A_5$  because leading order prediction of  $\theta_{13}$  gets sizable improvement as can be seen from Fig.

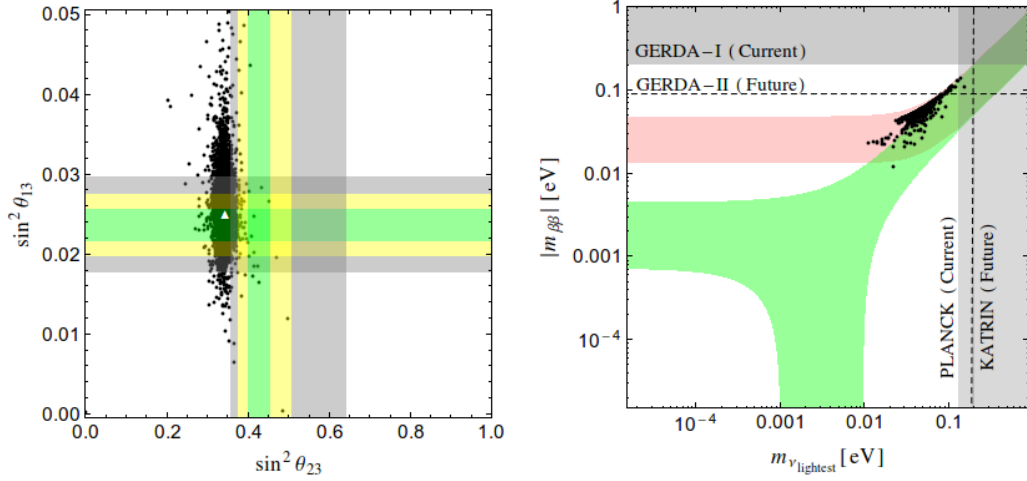


FIG. 3. Results of the perturbations due to the breaking of  $G_\nu$  in case of the group  $\Delta(6 \cdot 8^2)$ . All the points in left panel are obtained for  $|\epsilon_i| < 0.05$  in Eq. (30) and correspond to the values of  $\Delta m_{\text{solar}}^2$ ,  $\Delta m_{\text{atm}}^2$  and  $\theta_{12}$  in agreement with the global fits at  $3\sigma$ . Only points which fall in the  $3\sigma$  bands in the left panel are shown in the right panel. The green, yellow and gray bands in the left panel are the regions allowed at  $1\sigma$ ,  $2\sigma$  and  $3\sigma$  respectively by global fits. The white triangle is the prediction in the absence of perturbations. The green (red) band in the right panel corresponds to the most general allowed regions in case of normal (inverted) ordering in the neutrino masses. The gray bands are the regions ruled out by the current strongest limits from GERDA-I [43] and PLANCK and Galaxy clustering data [44] while the dashed lines correspond to the projected future limits by the ongoing experiments.

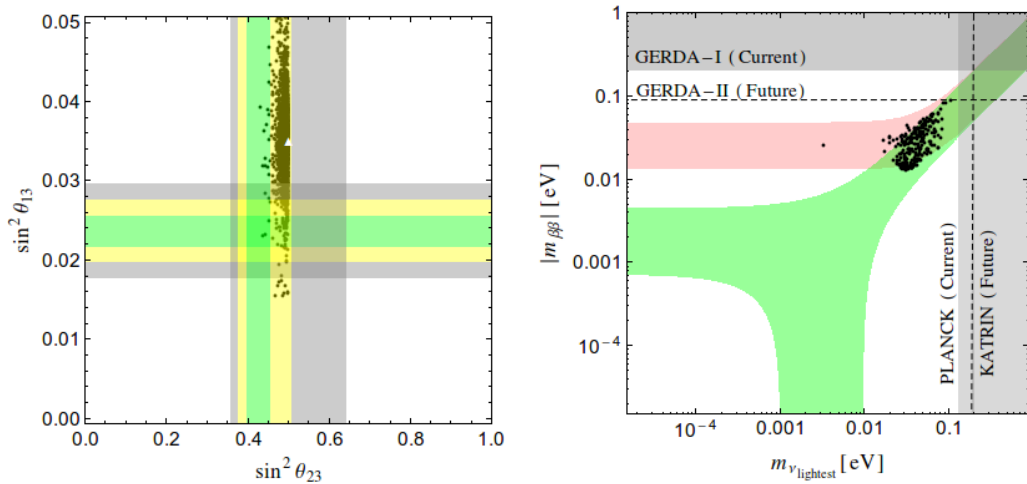


FIG. 4. Same as Fig. 3 but in the case of group  $A_5$ .

4. The corrections in  $\theta_{23}$  are however large enough to bring it in agreement with its global fit value at  $3\sigma$  in case of  $\Delta(6 \cdot 8^2)$  group. In the right panels, we show the predictions for

effective mass of neutrinoless double beta decay  $|m_{\beta\beta}|$  for the points in the left panels which reproduce both  $\theta_{13}$  and  $\theta_{23}$  in  $3\sigma$  ranges of their respective global fit values. As can be seen, all three neutrinos remain quasidegenerate in the small perturbation limit although original symmetry was designed to get only a degenerate solar pair. Most points in the figure fall in the region near to the sensitivity of the current generation experiments. Large perturbations corresponding to  $|\epsilon_i| > 0.05$  further strengthen the hierarchy in neutrino masses but lead to large deviations mainly in  $\theta_{13}$ .

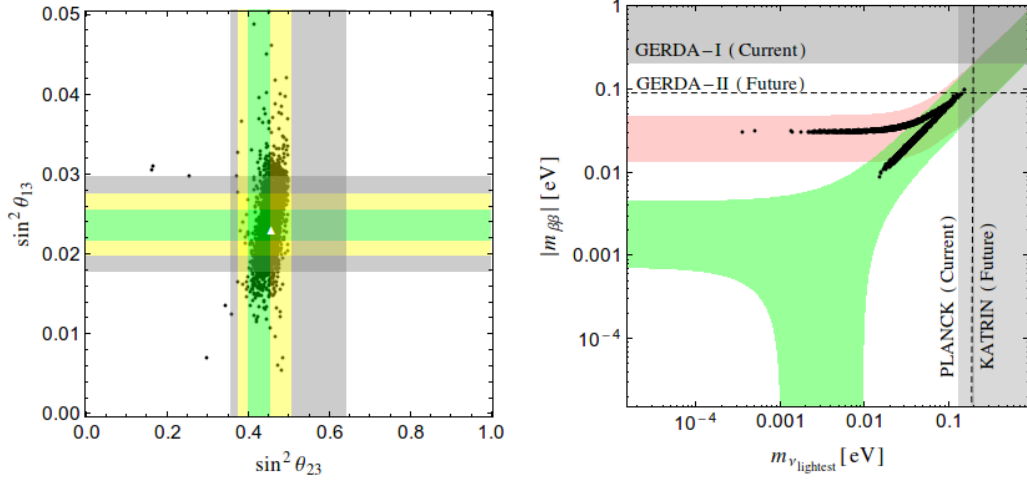


FIG. 5. Same as Fig. 3 but in the case of group  $PSL(2, 7)$ .

Similar results are shown in case of  $PSL(2, 7)$  in Fig. 5 for which both the reactor and atmospheric mixing angles are predicted within  $1\sigma$  at the leading order. Small perturbations can generate the solar mixing angle and  $\Delta m_{\text{solar}}^2$  without introducing large corrections in  $\theta_{23}$  and  $\theta_{13}$ . Further, the neutrino mass spectrum can be more hierarchical in this case compared to  $\Delta(6 \cdot 8^2)$  and  $A_5$ .

All three cases discussed above favor quasidegenerate neutrino spectrum and are consistent with both normal and inverted ordering in the neutrino masses. Unlike this, the solutions based on the DSG of  $U(3)$  favor strict inverted hierarchy at the leading order. Small breaking of the residual symmetry  $G_\nu$  can then generate small mass for third neutrino and the splitting in the solar pair. We discuss one such example based on the group  $\llbracket 324, 13 \rrbracket$  which offers the best predictions at the leading order among all the DSG of  $U(3)$ . One of the possible choice of the residual symmetries of neutrinos and charged leptons in this case is

$$S'_{1\nu} = \begin{pmatrix} 0 & 0 & \eta_{12} \\ 0 & \eta_{12} & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \text{and} \quad T'_l = - \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (32)$$

where  $\eta_{12} = e^{2\pi i/12}$ . The predicted leading order values of the mixing angles with this choice are given in Table II. The neutrino mass matrix with the above residual symmetry and

perturbed by the most general small corrections can be written as

$$M'_\nu = m_0 \begin{pmatrix} \epsilon_1 & 1 & \epsilon_4 \\ 1 & \epsilon_2 & \eta_{12}^2(1 + \epsilon_5) \\ \epsilon_4 & \eta_{12}^2(1 + \epsilon_5) & \epsilon_3 \end{pmatrix}. \quad (33)$$

Numerical analysis is performed for  $M'_\nu$  following the similar strategy described above and the results are displayed in Fig. 6. Unlike the examples based on the DSG of  $SU(3)$ , the

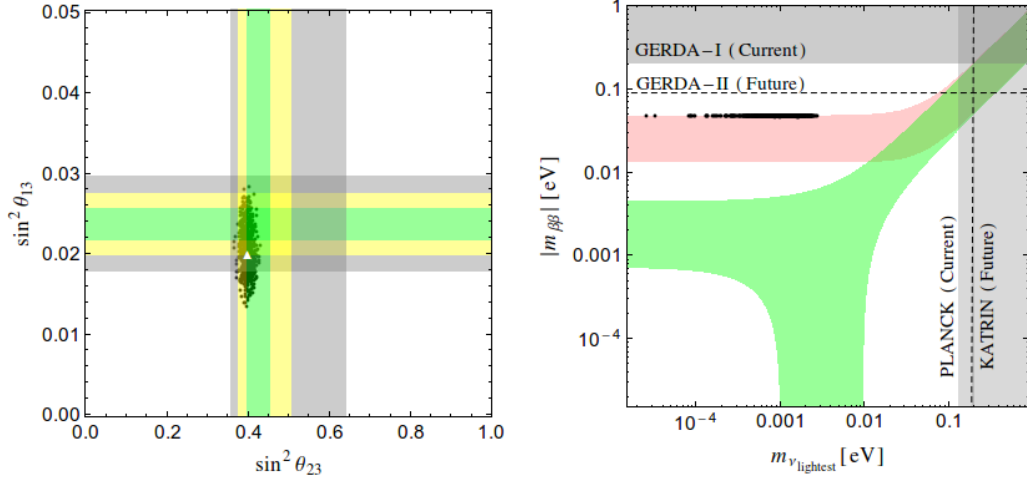


FIG. 6. Same as Fig. 3 but in the case of group  $[324,13]$ .

preference for the strong inverted hierarchy is clearly visible here. Apart from the solutions listed in the Table II, two the of the  $U(3)$  subgroups, namely  $[162,10]$  and  $[486,125]$ , lead to the  $\mu$ - $\tau$  symmetric  $M'_\nu$  and inverted hierarchy. The most general perturbations in this case are studied in [45]. It is found that small perturbations can produce viable corrections in  $\theta_{13}$  and  $\theta_{23}$  and both can be brought in to agreement with their global fit values in case of quasidegenerate and inverted hierarchy in neutrino masses but not for the normal hierarchy.

The CP violating phases  $\delta$ ,  $\beta_1$  do not remain arbitrary in all these cases once perturbations are introduced. We have determined them numerically and results are shown in Fig. 7. Only points which reproduce all the mixing angles and solar and atmospheric mass differences in their  $3\sigma$  ranges are shown. One clearly sees a preference for the maximal CP violation  $\delta = \pm \frac{\pi}{2}$  in all the cases except  $PSL(2,7)$  which allows larger deviation from it. We also estimate the invariant combinations of  $\theta_{12}$ ,  $\delta$  and  $\beta_1$  given in Eq. (10). We find in all the cases except  $PSL(2,7)$  that perturbations only introduce small corrections to their values predicted at the leading order given in Tables I, II. As a result, the phases shown in Fig. 7 are consistent with the phases inferred from the analytic expressions given in Eq. (10) and the leading order values of  $\mathcal{I}_e$  and  $\mathcal{I}_\mu$ . In the case of  $PSL(2,7)$ , the perturbations lead to significant corrections in CP phases from their values predicted at the leading order.

We discussed above the most general perturbation that may arise due to the breaking of  $G_\nu$  and can be parametrized in terms of four complex parameters  $\epsilon_i$  in Eq. (30) and five

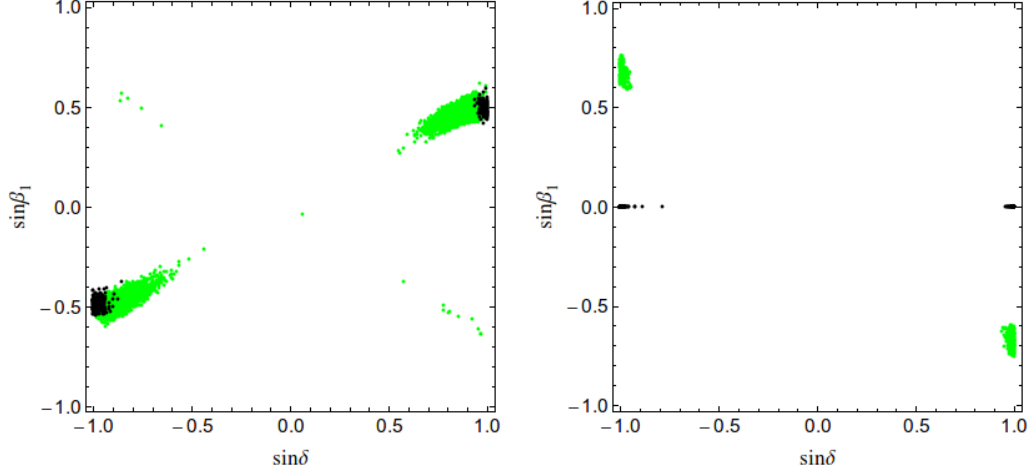


FIG. 7. Correlations between the Dirac CP phase  $\delta$  and Majorana phase  $\beta_1$  associated with quasidegenerate solar pair arising from the small perturbations (*i.e.*  $|\epsilon_i| \leq 0.05$ ) in different cases. The black (green) points in left panel correspond to the  $\Delta(6 \cdot 8^2)$  ( $PSL(2,7)$ ) while the same in the right panel correspond to the case  $A_5$  ( $[[324,13]]$ ). Only points that reproduce all three mixing angles in  $3\sigma$  ranges of global fit are shown.

in case of Eq. (33). Additional restrictions on  $\epsilon_i$  can be imposed if perturbation matrix  $\delta M_\nu = M_\nu - M_\nu^0$  is assumed to possess some symmetry. For example, following [25], it can be demanded that just like the leading order contribution, the perturbation also arises from the spontaneous breaking of  $G_f$ . Thus,  $\delta M_\nu$  is invariant under a different symmetry  $S_2$ . Both  $S_1$  and  $S_2$  are assumed to be subgroups of  $G_f$  so that introduction of appropriate flavon fields can lead to both the pieces after spontaneous breaking of  $G_f$ . Examples of such perturbations were given in [25] in the context of models discussed there. Let us give an economical example here for the group  $[[324,13]]$ . Consider the symmetry  $S_2$

$$S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \eta_{12}^{10} & 0 \\ 0 & 0 & \eta_{12}^{10} \end{pmatrix}. \quad (34)$$

This  $S_2$  is an element of the group  $[[324,13]]$  and can be expressed as  $S_2 = S^2(12, 3, 7, 3)$  in terms of one of the generators of the group  $[[324,13]]$  as given by Ludl in [37]. Only the  $(1, 1)$  element of the perturbation  $\delta M_\nu$  invariant under  $S_2$  is non-zero and the resulting  $M_\nu$  is thus obtained by putting  $\epsilon_{2,3,4,5} = 0$  in Eq. (33). The simple three parameter neutrino mass matrix so obtained can reproduce all the neutrino observables within  $1\sigma$  with the choice of parameters

$$m_0 = 0.0343 \text{ eV} \quad \text{and} \quad \epsilon_1 = 0.0366 - 0.0266 i.$$

## VII. SUMMARY

The recent progress in the field of discrete flavour symmetries suggest that such symmetries not only can lead to the specific flavour mixing pattern in the leptonic sector but also can be used to restrict the neutrino mass spectrum [24–26]. In this paper, we explored possible symmetries which can lead to the degenerate solar pair and a massive or massless third neutrino together with a realistic leptonic mixing pattern. Assuming that neutrinos are Majorana particles, it is shown that the DSG of  $SU(3)$  can lead to the former case while the latter can be achieved if the symmetry groups of the leptons are the DSG of  $U(3)$  and not  $SU(3)$ . The solar mixing angle becomes unphysical when the solar neutrinos are exactly degenerate. It is shown in this case that the symmetry groups determine the atmospheric and reactor mixing angles and certain combinations of the elements of lepton mixing matrix  $U$ , namely  $\text{Im}(U_{\alpha 1}^* U_{\alpha 2})$  with  $\alpha = e, \mu, \tau$ . From pure group theoretical considerations and utilizing the available information about the DSG of  $U(3)$ , we carry out the detailed analysis for possible predictions of mixing angles and degenerate solar neutrinos at the leading order. The study presented here encompasses all the classified DSG of  $SU(3)$  having three dimensional IR. Our main observations are summarized in the following.

- None of the  $SU(3)$  subgroups of type C leads to the realistic predictions for  $\theta_{23}$  and  $\theta_{13}$  simultaneously.
- All the  $SU(3)$  subgroups of type D predicts  $\sin^2 \theta_{23} \cos^2 \theta_{13} = 1/3$ . In this category, all the groups in  $\Delta(6n^2)$  series for  $n \geq 8$  (except  $10 \leq n \leq 14$  and  $n = 20, 21$ ) can predict  $\theta_{13}$  within the  $3\sigma$  range of global fit. The prediction for  $\theta_{13}$  by the other remaining groups in this category is subset of the predictions given by the  $\Delta(6n^2)$  groups.
- Among the DSG of  $SU(3)$  which are not in category C or D, the group  $PSL(2, 7) \cong \Sigma(168)$  predicts both  $\theta_{23}$  and  $\theta_{13}$  very near to their best fit values.
- Among all the DSG of  $U(3)$  of order  $< 512$ , only the group  $\llbracket 324, 13 \rrbracket$  predicts both  $\theta_{13}$  and  $\theta_{23}$  simultaneously in their  $3\sigma$  ranges. This symmetry can lead to a degenerate solar pair and a massless third neutrino as a leading order realization of inverted hierarchy.

Suitable corrections to the above predictions are needed to generate the viable solar mass difference and solar mixing angle. They may arise in complete models due to the breaking of residual symmetries of neutrinos. The examples of such most general perturbations are discussed in the case of  $\Delta(6 \cdot 8^2)$ ,  $A_5$ ,  $PSL(2, 7)$  and  $\llbracket 324, 13 \rrbracket$  groups and it is shown that small perturbations can generate viable solar mass difference and all three mixing angles in agreement with their global fit values. If perturbations are small, all three cases based on DSG of  $SU(3)$  show preference for quasidegenerate neutrino spectrum while the group  $\llbracket 324, 13 \rrbracket$  predicts strong inverted hierarchy. All these cases except  $PSL(2, 7)$  also lead to nearly maximal CP violating phase  $\delta$ .

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